

A CONSTRUCTION OF A GENERALISED QUANTUM SWAP GATE

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ABSTRACT. Most often it is assumed that a quantum computer is predicated on a collection of two-dimensional quantum mechanical systems called qubits that acts on a quantum circuit. The qubit SWAP gate has been illustrated to be a cornerstone in the networkability of quantum computer. However, there is a view to generalise quantum computing to a collection of d -dimensional, or qudit, quantum mechanical systems. We present a construction that generalises the qubit SWAP gate to quantum systems based on qudits over prime dimensions.

1. INTRODUCTION

Of central importance to the theory of quantum computation is the role assumed by multiple qubit gates in establishing a basis for quantum circuitry designs. A quantum circuit is an assembly of discrete sets of components which describe computational procedures (Nielsen and Chuang (2000a)). Physical implementations of such designs describe the process of computation whereby the evolution of a quantum state and its influence on other states can be modelled. In many aspects, the quantum network approach to computation resembles the classical procedure to computing (Vlasov (2004)) where quantum circuits are formed from a composition of quantum states, quantum gates and quantum wires (Nielsen and Chuang (2000b)). Our ability to preserve the coherence of quantum state rests with our ability to implement quantum computations successfully. Quantum computations are described within the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ of n qubits. Horizontal quantum circuit wires correspond to the individual \mathbb{C}^2 subspaces. Vertical wires in a quantum circuit represent the *coupling* of arbitrary pairs of quantum gates, and, as such, quantum computations identify the changes imposed on a quantum state during the implementation of quantum gates in a manner analogous to classical implementation of logic gates. Those quantum gates that have been experimentally demonstrated are said to be elements of the quantum gate library. Unfortunately, there are only a handful of quantum gates that can be experimentally realised within the coherence time of their systems (Vatan and Williams (2004)). Barenco *et al.* (1995) showed that any quantum gate on a set of n -qubits can be restricted to a composition of realisable gates; the controlled-NOT (CNOT) gate and single qubit gates. For this reason, we say that the qubit gate library consisting of single qubit gates and the CNOT gate is universal, and in doing so, it has become standard in quantum information to express any n -qubit quantum operation as a composition of single qubit gates and CNOT gates.

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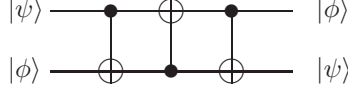


FIGURE 1. Quantum circuit swapping two qubits.

Most often it is assumed that a quantum computer is predicated on a collection of two-dimensional quantum mechanical systems called qubits. However, there has been the view to generalise to d -dimensional, or *qudit*, quantum mechanical systems. Given an arbitrary finite alphabet Σ of cardinality d , we process quantum information by specifying a state description of a finite dimension quantum space - in particular, the state description of the Hilbert space \mathbb{C}^d . While the state of an d -dimensional Hilbert space can be more generally expressed as a linear combination of basis states $|\psi_i\rangle$, we write each orthonormal basis state of the d -dimensional Hilbert space \mathbb{C}^d to correspond with an element of \mathbb{Z}_d . In this context the basis $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ is referred to as the *computational basis*. Therefore, a state $|\psi\rangle$ of \mathbb{C}^d is given by $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$, where $\alpha_i \in \mathbb{C}$ and $\sum_{i=0}^{d-1} |\alpha_i|^2 = 1$. A qudit describes a state in the Hilbert space \mathbb{C}^d , and the state space of an n -qudit state is the tensor product of the basis states of the single system \mathbb{C}^d , written $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, with corresponding orthonormal basis states given by $|i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_n\rangle = |i_1 i_2 \dots i_n\rangle$, where $i_j \in \mathbb{Z}_d$. The general state of a qudit in the Hilbert space \mathcal{H} is then written

$$(1.1) \quad |\psi\rangle = \sum_{(i_1 i_2 \dots i_n) \in \mathbb{Z}_d^n} \alpha_{(i_1 i_2 \dots i_n)} |i_1 i_2 \dots i_n\rangle,$$

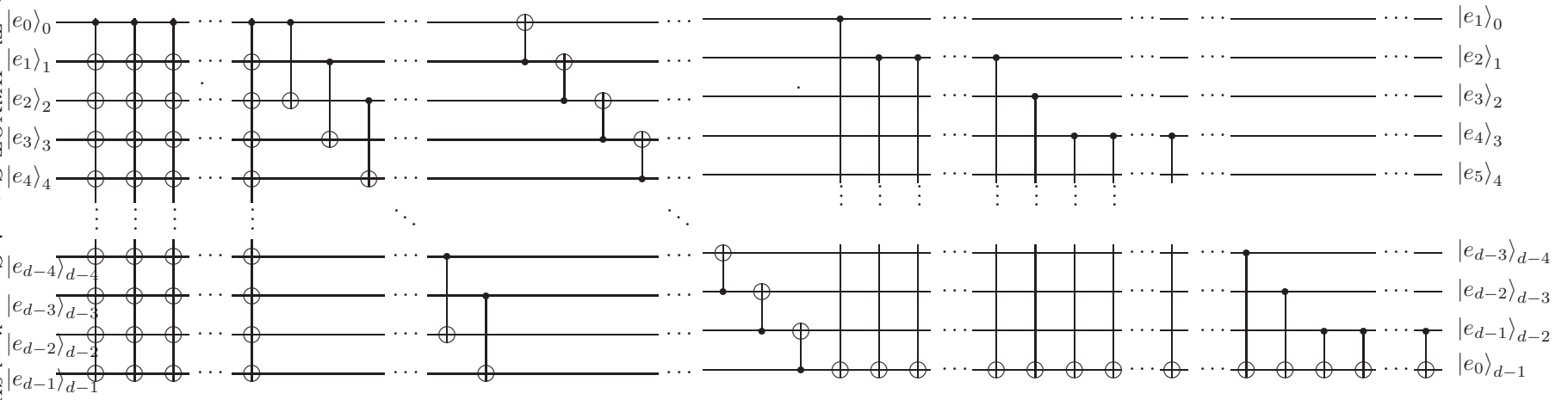
where $\alpha_{(i_1 i_2 \dots i_n)} \in \mathbb{C}$ and $\sum |\alpha_{(i_1 i_2 \dots i_n)}|^2 = 1$. The qudit representation of a quantum state provides a natural mechanism by which quantum computations can be implemented. That such a computation is made possible initially lies with the notion of state signature. In particular, the correspondence of quantum information α_k with a computational qudit basis element $|k\rangle$ and the subsequent genesis of the quantum state $\sum_{k=0}^{d-1} \alpha_k |k\rangle$ in the Hilbert space \mathbb{C}^d . Such a correspondence between information and a Hilbert space representation is prerequisite to quantum computation since the successful transmission of any information state is predicated on encoding the basis states associated with the quantum information elements rather than the information itself.

Let \mathcal{H}_A and \mathcal{H}_B be two d -dimensional Hilbert spaces with bases $|i\rangle_A$ and $|i\rangle_B, i \in \mathbb{Z}_d$ respectively. Let $|\psi\rangle_A$ denote a pure state of the quantum system \mathcal{H}_A . Similarly, let $|\phi\rangle_B$ denote a pure state of the quantum system \mathcal{H}_B and consider an arbitrary unitary transformation $U \in U(d^2)$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. Let U_{CNOT} denote a controlled-NOT (CNOT) gate that has qudit $|\psi\rangle_A$ as the control qudit and $|\phi\rangle_B$ as the target qudit; then

$$(1.2) \quad U_{\text{CNOT}} |m\rangle_A \otimes |n\rangle_B = |m\rangle_A \otimes |n \oplus m\rangle_B, \quad m, n \in \mathbb{Z}_d$$

where $i \oplus j$ denote modulo d addition. We now introduce a quantum gate construction - determined entirely from instances of the CNOT gate - that generalises the qubit SWAP gate (see Fig. 1) to higher dimensional quantum systems.

FIGURE 2. The WII NOT Gate; A Generalised SWAP Gate.



2. THE WILNOT GATE

The WilNOT gate is a generalised quantum SWAP gate (see Fig. 2) that cycles the states of d d -dimensional quantum systems. Suppose that the first quantum system \mathcal{A}_0 prepared in the state $|e_0\rangle_0$, the second system \mathcal{A}_1 prepared in the state $|e_1\rangle_1$ and so forth, with the final system \mathcal{A}_{d-1} prepared in the state $|e_{d-1}\rangle_{d-1}$. Construction of the WilNOT gate over prime dimension yields a generalised SWAP gate so that the system \mathcal{A}_0 is in the state $|e_1\rangle_0$, the system \mathcal{A}_1 is in the state $|e_2\rangle_1$ and so forth, until the system \mathcal{A}_{d-1} is in the state $|e_0\rangle_{d-1}$. Central to this implementation is the use of the generalised quantum controlled-NOT operator, $|x\rangle|y\rangle \mapsto |x\rangle|y \oplus x \bmod d\rangle$.

Lemma 2.1. (*Rosen (2000)*) $\sum_{n=0}^k \binom{l+n}{n} = \binom{l+k+1}{k}$.

Theorem 2.2. Let $d = p$ be a prime. The WilNOT operator algorithm provides a construction for a generalised quantum SWAP operator through uses of the generalised quantum controlled-NOT operator. The quantum SWAP operator has

Input: $|e_k\rangle_k; k = 0, \dots, d-1$.

Output: $|e_{k+1}\rangle_k; k = 0, \dots, d-2, |e_0\rangle_{d-1}$.

The WilNOT operator algorithm is described as follows:

Input: $e_k := i_k^0; k = 0, \dots, d-1$

Output: $i_k^{d+2} = e_{k+1}; k = 0, \dots, d-1, i_{d-1}^{d+2} = e_0$

Stage 1: Initialisation $j = 0$.

$$e_k := i_k^0$$

for $k = 0, \dots, d-1$. The WilNOT gate is initiated by Stage 1 and step $j = 0$ by making the correspondence between a representative input element i_k^0 of the WilNOT gate algorithm and each standard basis state e_k .

Stage 2: For $j = 1, \dots, d-1$.

$$\begin{aligned} i_0^j &= i_0^{j-1} \\ i_k^j &= i_{k-1}^j + i_k^{j-1}; k = 1, \dots, d-1. \end{aligned}$$

Stage 2 consists of $d-1$ steps which repeat the sequence of gates of step $j = 1$. The sequence of gates at step $j = 1$, see Fig. 3, is targeted on the systems $\mathcal{A}_1, \dots, \mathcal{A}_{d-1}$. Each step of Fig. 3 is a composition of CNOT gates acting on consecutive pairs of systems and is written as a shorthand form to represent the sequence of CNOT gates given in Fig. 4. The algorithm process of step $j = 1$ transforms the input sequence $i_0^0, i_1^0, i_2^0, \dots, i_{d-1}^0$ to the resulting state given by $i_0^0, \sum_{k=0}^1 i_k^0, \sum_{k=0}^2 i_k^0, \dots, \sum_{k=0}^{d-1} i_k^0$. In a similar manner, the WilNOT gate at step $j = 2$ takes the output from step $j = 1$ as input and repeats the sequence of gates. The resulting state of the circuit at step $j = 2$ is given by $i_0^0, i_0^0 + \sum_{k=0}^1 i_k^0, i_0^0 + \sum_{k=0}^1 i_k^0 + \sum_{k=0}^2 i_k^0, \dots, i_0^0 + \sum_{k=0}^1 i_k^0 + \sum_{k=0}^2 i_k^0 + \dots + \sum_{k=0}^{d-1} i_k^0$. This process continues to step $j = d-1$. Figure 5 illustrates initialisation on the circuit and the subsequent $d-1$ steps of Stage 2.

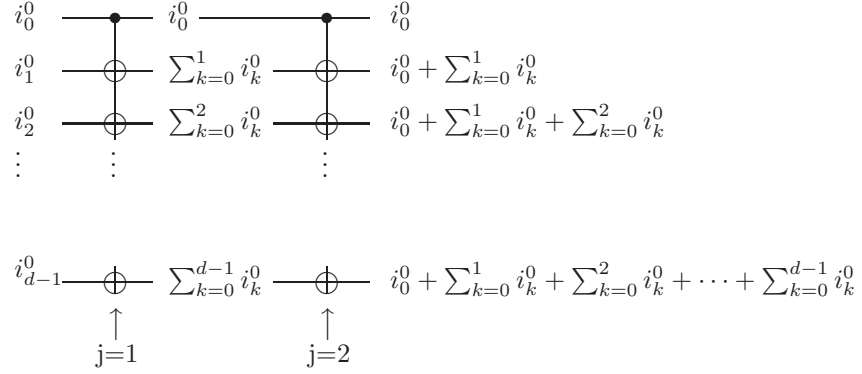
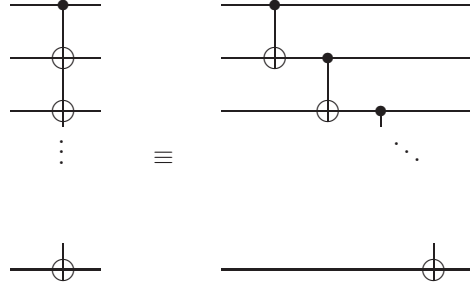
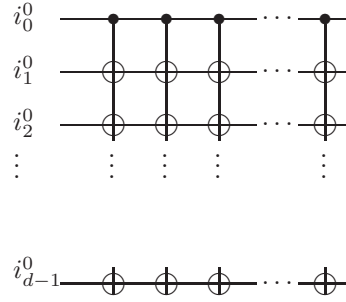
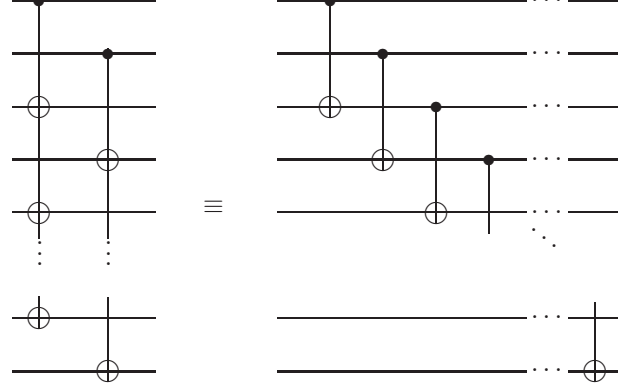
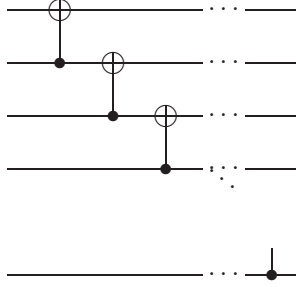
FIGURE 3. WilNOT gate; Stage 2, steps $j = 1, 2$.

FIGURE 4. WilNOT gate; Stage 2. Algorithm step operates on successive pairs.

FIGURE 5. WilNOT gate; Stage 2, steps $j = 1, \dots, d-1$.

Stage 3: $j = d$.

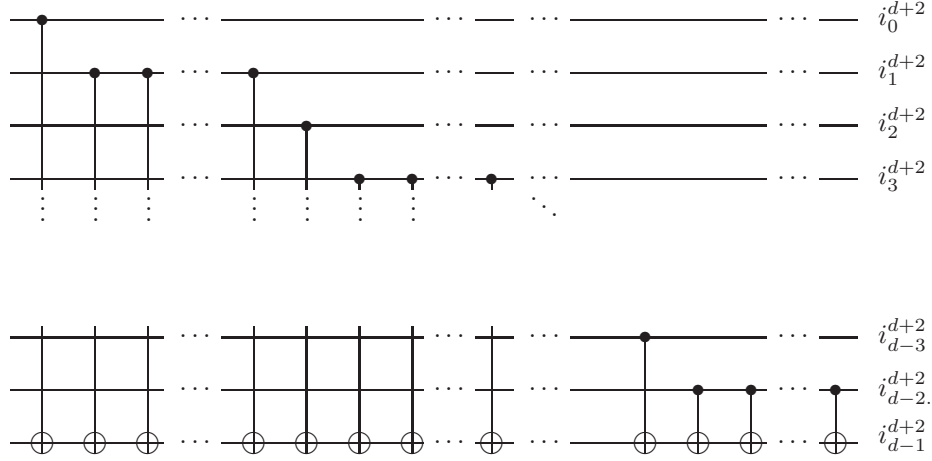
$$\begin{aligned}
 i_0^d &= i_0^{d-1} \\
 i_1^d &= i_1^{d-1} \\
 i_k^d &= i_{k-2}^d + i_k^{d-1}; k = 2, \dots, d-1.
 \end{aligned}$$

FIGURE 6. WilNOT gate; Stage 3, step $j = d$.FIGURE 7. WilNOT gate; Stage 4, step $j = d + 1$.

The sequence of values $i_0^{d-1}, i_1^{d-1}, \dots, i_{d-1}^{d-1}$ corresponding to the final step of Stage 2 are carried forward as an input sequence for Stage 3 and step $j = d$. The algorithm step keeps the values i_0^{d-1}, i_1^{d-1} and returns them as outcomes i_0^d, i_1^d for step $j = d$. The remaining systems are then targeted in an iterative process. For instance, the outcome i_2^d for step $j = d$ is given by $i_0^d + i_2^{d-1}$. This value is then stored as the result i_2^d for e_2 at Stage 3. The outcome state for $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ at Stage 3 have thus been determined. To evaluate the result value for \mathcal{A}_3 , the algorithm computes $i_1^d + i_3^{d-1}$ and stores this value as the outcome i_3^d for Stage 3. Fig. 6 illustrates the process that determines the current state of the algorithm following stage 3 and step $j = d$ in diagrammatic shorthand form for the sequence of CNOTs.

Stage 4: $j = d + 1$.

$$\begin{aligned} i_k^{d+1} &= i_k^d + i_{k+1}^d \\ i_{d-1}^{d+1} &= i_{d-1}^d; k = 0, \dots, d-2. \end{aligned}$$

FIGURE 8. WilNOT gate; Stage 5, step $j = d + 2$.

Stage 4 consists of a single step, $j = d + 1$, whose primary algorithm operation acts as a CNOT on the $d - 1$ consecutive pairs of systems $(\mathcal{A}_k, \mathcal{A}_{k+1})$ for $k = 0, \dots, d - 2$, computing $(i_k^d + i_{k+1}^d)$ and storing these values as the outcome i_k^{d+1} . The value i_{d-1}^d is returned as the outcome i_{d-1}^{d+1} .

Stage 5: $j = d + 2$.

$$\begin{aligned} i_k^{d+2} &= i_k^{d+1} \\ i_{d-1}^{d+2} &= i_{d-1}^{d+1} + \sum_{k=0}^{d-2} \eta_k i_k^{d+2}; \quad k = 0, \dots, d - 2 \end{aligned}$$

with

$$(2.1) \quad \sum_{k=0}^{d-2} \eta_k i_k^{d+2} := \sum_{t=0}^{\lfloor \frac{d-2}{2} \rfloor} (d-1) i_{2t+1}^{d+2} + \sum_{t=0}^{\frac{d-3}{2}} i_{2t}^{d+2}.$$

Stage 5 concludes the WilNOT gate transformation with a set of gates targeted on system \mathcal{A}_{d-1} whose current state is represented by i_{d-1}^{d+1} . The values $i_0^{d+2}, i_1^{d+2}, \dots, i_{d-2}^{d+2}$ for the respective systems $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{d-2}$ are unchanged from their representative values $i_0^{d+1}, i_1^{d+1}, \dots, i_{d-2}^{d+1}$ at step $j = d + 1$ and are returned as outcomes in the final state for step $j = d + 2$. The final state of \mathcal{A}_{d-1} is given by $i_{d-1}^{d+2} = i_{d-1}^{d+1} + \sum_{k=0}^{d-2} \eta_k i_k^{d+2} = i_{d-1}^{d+1} + i_0^{d+1} + (d-1)i_1^{d+1} + i_2^{d+1} + (d-1)i_3^{d+1} + \dots + i_{d-3}^{d+1} + (d-1)i_{d-2}^{d+1}$. Thus, for odd valued k there is a gate with i_k^{d+2} as control and for even valued k there are $d-1$ gates with i_k^{d+2} as control. This is represented in Fig. 8.

Proof: We show that the algorithm outputs $i_k^{d+2} = e_{k+1}$ for $k = 0, \dots, d + 2$ and $i_{d-1}^{d+2} = e_0$. At step $j = 0$, we have it that,

$$e_k := i_k^0; \quad k = 0, \dots, d - 1.$$

At Stage 2, step $j = 1$ the algorithm sets $i_1^1 = i_0^0$ and computes i_1^1 as $i_1^1 = i_1^0 + i_0^0$. Similarly, $i_2^1 = i_1^0 + i_2^0 = i_0^0 + i_1^0 + i_2^0 = \sum_{m=0}^2 i_m^0$. Therefore, for $k = 1, \dots, d - 1$,

we have,

$$\begin{aligned}
 i_k^1 &= i_{k-1}^1 + i_k^0 \\
 &= i_{k-2}^1 + i_{k-1}^0 + i_k^0 \\
 &= i_{k-3}^1 + i_{k-2}^0 + i_{k-1}^0 + i_k^0 \\
 &= \dots \\
 &= i_0^1 + i_1^0 + i_2^0 + \dots + i_{k-3}^0 + i_{k-2}^0 + i_{k-1}^0 + i_k^0 \\
 (2.2) \quad &= \sum_{m=0}^k i_m^0.
 \end{aligned}$$

The next step, Stage 2 step $j = 2$, implements a repeat set of gates of step 1. By definition $i_0^2 = i_0^1 = i_0^0$. The case for $k = 1, \dots, d-1$ follows from the algorithm step,

$$\begin{aligned}
 i_k^2 &= i_{k-1}^2 + i_k^1 \\
 &= i_{k-2}^2 + i_{k-1}^1 + i_k^1 \\
 &= \dots \\
 &= i_0^2 + i_1^1 + i_2^1 + \dots + i_{k-2}^1 + i_{k-1}^1 + i_k^1 \\
 &= i_0^0 + \sum_{m=0}^1 i_m^0 + \sum_{m=0}^2 i_m^0 + \dots + \sum_{m=0}^k i_m^0 \\
 &= \sum_{l=0}^k \sum_{m=0}^l i_m^0 \\
 &= \sum_{m=0}^k \sum_{l=m}^k i_m^0 \\
 &= \sum_{m=0}^k \sum_{l=0}^{k-m} i_m^0 \\
 (2.3) \quad &= \sum_{m=0}^k \binom{k-m+1}{1} i_m^0.
 \end{aligned}$$

We show by induction that, for $j = 1, \dots, d-1$, $i_k^j = \sum_{m=0}^k \binom{k-m+j-1}{j-1} i_m^0$, $k = 0, \dots, d-1$. We have shown that this is true for $j = 1$. Let $1 \leq j < d-1$ and suppose that

$$(2.4) \quad i_k^j = \sum_{m=0}^k \binom{k-m+j-1}{j-1} i_m^0,$$

$k = 0, \dots, d-1$. Now, $i_0^{j+1} = i_0^j = i_0^0$. For $1 \leq k \leq d-1$, we have

$$\begin{aligned}
 i_k^{j+1} &= i_k^j + i_{k-1}^{j+1} \\
 &= i_k^j + i_{k-1}^j + i_{k-2}^{j+1} \\
 &= \dots \\
 &= i_k^j + i_{k-1}^j + \dots + i_2^j + i_1^j + i_0^{j+1} \\
 (2.5) \quad &= i_k^j + i_{k-1}^j + \dots + i_2^j + i_1^j + i_0^j.
 \end{aligned}$$

Since $i_0^{j+1} = i_0^j$ follows from the algorithm step, we have it that $i_k^{j+1} = \sum_{m=0}^k i_m^j$. Hence, by the induction process,

$$\begin{aligned}
 i_k^{j+1} = \sum_{m=0}^k i_m^j &= i_0^0 + \sum_{l=0}^1 \binom{1-l+j-1}{j-1} i_l^0 + \sum_{l=0}^2 \binom{2-l+j-1}{j-1} i_l^0 + \dots \\
 &\quad + \sum_{l=0}^k \binom{k-l+j-1}{j-1} i_l^0 \\
 &= \sum_{m=0}^k \sum_{l=0}^m \binom{m-l+j-1}{j-1} i_l^0 \\
 &= \sum_{m=0}^k \sum_{l=0}^k \binom{m-l+j-1}{j-1} i_l^0 \\
 &= \sum_{l=0}^k \sum_{m=l}^k \binom{m-l+j-1}{j-1} i_l^0 \\
 (2.6) \quad &= \sum_{l=0}^k \binom{k-l+j}{j} i_l^0.
 \end{aligned}$$

Therefore, the result is true for $j+1$

$$(2.7) \quad i_k^{j+1} = \sum_{m=0}^k \binom{k-m+j}{j} i_m^0$$

and the result follows by induction.

The algorithm at Stage 3, step $j = d$ gives

$$\begin{aligned}
 i_0^d &= i_0^{d-1} = i_0^0 \\
 (2.8) \quad i_1^d &= i_1^{d-1} = \sum_{m=0}^1 \binom{1-m+d-2}{d-2} i_m^0 = (d-1)i_0^0 + i_1^0.
 \end{aligned}$$

Implementing the algorithm step $i_k^d = i_{k-2}^d + i_k^{d-1}$ for $k = 2, \dots, d-1$, we have it that

$$\begin{aligned}
 i_2^d &= i_0^d + i_2^{d-1} = i_0^0 + \sum_{m=0}^2 \binom{2-m+d-2}{d-2} i_m^0 \\
 i_3^d &= i_1^d + i_2^{d-1} = \sum_{m=0}^1 \binom{1-m+d-2}{d-2} i_m^0 + \sum_{m=0}^3 \binom{3-m+d-2}{d-2} i_m^0 \\
 i_4^d &= i_2^d + i_4^{d-1} = i_0^0 + \sum_{m=0}^2 \binom{2-m+d-2}{d-2} i_m^0 + \sum_{m=0}^4 \binom{4-m+d-2}{d-2} i_m^0 \\
 &\dots
 \end{aligned}$$

Therefore, for odd valued k ,

$$\begin{aligned}
 i_k^d &= \sum_{t=0}^{\frac{k-1}{2}} \sum_{m=0}^{2t+1} \binom{2t+1-m+d-2}{d-2} i_m^0 \\
 &= \sum_{t=0}^{\frac{k-1}{2}} \sum_{m=2t}^{2t+1} \binom{2t+1-m+d-2}{d-2} i_m^0 \quad (\text{as } d \text{ is prime}) \\
 (2.9) \quad &= \sum_{t=0}^{\frac{k-1}{2}} (d-1) i_{2t}^0 + i_{2t+1}^0
 \end{aligned}$$

and, similarly, for even valued k ,

$$\begin{aligned}
 i_k^d &= \sum_{t=0}^{\frac{k}{2}} \sum_{m=0}^{2t} \binom{2t-m+d-2}{d-2} i_m^0 \\
 &= \sum_{t=0}^{\frac{k}{2}} \sum_{m=2t}^{2t} \binom{2t-m+d-2}{d-2} i_m^0 \quad (\text{as } d \text{ is prime}) \\
 (2.10) \quad &= \sum_{t=0}^{\frac{k}{2}} (d-1) i_{2t-1}^0 + i_{2t}^0.
 \end{aligned}$$

Stage 4, step $j = d + 1$ of the algorithm is given by $i_k^{d+1} = i_k^d + i_{k+1}^d$ for $k = 0, \dots, d-2$. Let us consider i_k^{d+1} . There are two cases; for even valued k , we note that

$$\begin{aligned}
 i_k^d &= i_{k-2}^d + i_k^{d-1} \\
 &= i_{k-4}^d + i_{k-2}^{d-1} + i_k^{d-1} \\
 &= \dots \\
 (2.11) \quad &= \sum_{t=0}^{\frac{k}{2}} i_{2t}^{d-1}
 \end{aligned}$$

while

$$\begin{aligned}
 i_{k+1}^d &= i_{k-1}^d + i_{k+1}^{d-1} \\
 &= i_{k-3}^d + i_{k-1}^{d-1} + i_{k+1}^{d-1} \\
 &= \dots \\
 (2.12) \quad &= \sum_{t=0}^{\lfloor \frac{k+1}{2} \rfloor} i_{2t+1}^{d-1}.
 \end{aligned}$$

Therefore, $i_k^{d+1} = \sum_{t=0}^{k+1} i_t^{d-1}$ for even valued k . Alternatively, for odd valued k then $i_k^d = \sum_{t=0}^{\frac{k-1}{2}} i_{2t+1}^{d-1}$ while $i_{k+1}^d = \sum_{t=0}^{\frac{k+1}{2}} i_{2t}^{d-1}$ and $i_k^{d+1} = \sum_{t=0}^{k+1} i_t^{d-1}$. Hence,

$$\begin{aligned}
 i_k^{d+1} &= \sum_{t=0}^{k+1} i_t^{d-1} \\
 &= \sum_{l=0}^{k+1} \sum_{m=0}^t \binom{t-m+d-2}{d-2} i_m^0 \\
 &= \sum_{m=0}^{k+1} \sum_{l=m}^{k+1} \binom{l-m+d-2}{d-2} i_m^0 \\
 &= \sum_{m=0}^{k+1} \binom{k-m+d}{d-1} i_m^0 \\
 (2.13) \quad &= i_{k+1}^0 \pmod{d}.
 \end{aligned}$$

Recall that since the dimension, $d = p$, considered is prime, under arithmetic modulo d , $\binom{k-m+d}{d-1}$ vanishes for $m \neq k+1$ and therefore we deduce that $i_k^{d+1} = i_{k+1}^0$ for $k = 0, \dots, d-2$. When $k = d-1$, we have

$$(2.14) \quad i_{d-1}^{d+1} = i_{d-1}^d = \sum_{t=0}^{\frac{d-1}{2}} \sum_{m=0}^{2t} \binom{2t-m+d-2}{d-2} i_m^0.$$

The WilNOT gate concludes at Stage 5, step $j = d+2$ with the implementation of a sequence of gates targeted on i_{d-1}^{d+1} . For $k = 0, \dots, d-2$, we have the result

$$(2.15) \quad i_k^{d+2} = i_k^{d+1} = i_{k+1}^0 \pmod{d}.$$

For $k = d-1$, the value of i_{d-1}^{d+2} is given by

$$\begin{aligned}
 i_{d-1}^{d+2} &= i_{d-1}^{d+1} + \sum_{k=0}^{d-2} \eta_k i_k^{d+2} \\
 &= \sum_{t=0}^{\frac{d-1}{2}} \sum_{m=0}^{2t} \binom{2t-m+d-2}{d-2} i_m^0 + \sum_{k=0}^{d-2} \eta_k i_k^{d+2}.
 \end{aligned}$$

To show that this returns the desired result, we consider the value $i_{d-1}^{d+1} \pmod{d}$.

Lemma 2.3. $i_{d-1}^{d+1} \pmod{d} = \sum_{t=0}^{\frac{d-1}{2}} i_{2t}^0 + \sum_{t=0}^{\frac{d-1}{2}-1} (d-1) i_{2t+1}^0$.

Proof.

$$\begin{aligned}
 i_{d-1}^{d+1} &= \sum_{t=0}^{\frac{d-1}{2}} \sum_{m=0}^{2t} \binom{2t-m+d-2}{d-2} i_m^0 \\
 (2.16) \quad &= \sum_{m=0}^{d-1} \sum_{t=\lceil \frac{m}{2} \rceil}^{\frac{d-1}{2}} \binom{2t-m+d-2}{d-2} i_m^0.
 \end{aligned}$$

Since $\binom{2t-m+d-2}{d-2} = 0 \pmod d$ for $t > \lceil \frac{m}{2} \rceil$ then

$$\begin{aligned}
 i_{d-1}^{d+1} \pmod d &= \sum_{m=0}^{d-1} \binom{2\lceil \frac{m}{2} \rceil - m + d - 2}{d-2} i_m^0 \\
 (2.17) \qquad &= \sum_{l=0}^{\frac{d-1}{2}} i_{2l}^0 + \sum_{l=0}^{\lfloor \frac{d-2}{2} \rfloor} (d-1) i_{2l+1}^0.
 \end{aligned}$$

Thus, $i_{d-1}^{d+1} \pmod d = \sum_{t=0}^{\frac{d-1}{2}} i_{2t}^0 + \sum_{t=0}^{\lfloor \frac{d-2}{2} \rfloor} (d-1) i_{2t+1}^0$. By definition of Stage 5, we have it that $\sum_{k=0}^{d-2} \eta_k i_k^{d+2} = \sum_{t=0}^{\lfloor \frac{d-2}{2} \rfloor} (d-1) i_{2t+1}^{d+2} + \sum_{t=0}^{\frac{d-3}{2}} i_{2t}^{d+2}$. The value of i_{d-1}^{d+2} is then given by

$$\begin{aligned}
 i_{d-1}^{d+2} &= i_{d-1}^{d+1} + \sum_{k=0}^{d-2} \eta_k i_k^{d+2} \\
 (2.18) \quad &= \sum_{t=0}^{\lfloor \frac{d-1}{2} \rfloor} i_{2t}^0 + \sum_{t=0}^{\lfloor \frac{d-2}{2} \rfloor} (d-1) i_{2t+1}^0 + \sum_{t=0}^{\lfloor \frac{d-2}{2} \rfloor} i_{2t+1}^0 + \sum_{t=1}^{\lfloor \frac{d-1}{2} \rfloor} (d-1) i_{2t}^0.
 \end{aligned}$$

Consequently, $i_{d-1}^{d+2} \pmod d = i_0^{d+2} = i_0^0$. Stage 5 of the algorithm ensures that the WilNOT gate effectuates the transformation of an input sequence given by $i_k^0 = e_k$ for $k = 0, \dots, d-1$ to the sequence $i_k^{d+2} = e_{k+1}$ for $k = 0, \dots, d-2$ and $i_{d-1}^{d+2} = e_0$, thereby finalising the construction process for a generalised quantum SWAP gate. We show that the network swaps all d^d sequences of input states.

Theorem 2.4. *Let $\mathcal{A}_0, \dots, \mathcal{A}_{d-1}$ be d -dimensional systems with bases $|e_0\rangle_j, |e_1\rangle_j, \dots, |e_{d-1}\rangle_j$, $j = 0, \dots, d-1$, where $e_0, \dots, e_{d-1} \in \mathbb{Z}_d$. Let $\mathcal{A} = \mathcal{A}_0 \otimes \dots \otimes \mathcal{A}_{d-1}$. If a network implements a SWAP on each basis state $|a_0 a_1 \dots a_{d-1}\rangle = |a_0\rangle_0 \otimes |a_1\rangle_1 \otimes \dots \otimes |a_{d-1}\rangle_{d-1}$ of \mathcal{A} where $a_0, \dots, a_{d-1} \in \mathbb{Z}_d$ then the network implements a SWAP on any input state $|\psi\rangle = |\psi_0\rangle_0 \otimes |\psi_1\rangle_1 \otimes \dots \otimes |\psi_{d-1}\rangle_{d-1}$.*

Proof: Let $|\psi_j\rangle_j = \sum_{k_j=0}^{d-1} \alpha_{jk_j} |e_{k_j}\rangle_j$, $j = 0, \dots, d-1$. Then

$$(2.19) \quad |\psi\rangle = \sum_{k_0=0}^{d-1} \dots \sum_{k_{d-1}=0}^{d-1} \alpha_{0k_0} \dots \alpha_{(d-1)k_{d-1}} |k_0 \dots k_{d-1}\rangle.$$

Now,

$$\begin{aligned}
 \text{SWAP}(|\psi\rangle) &= \sum_{k_0=0}^{d-1} \dots \sum_{k_{d-1}=0}^{d-1} \alpha_{0k_0} \dots \alpha_{(d-1)k_{d-1}} \text{SWAP}(|k_0 \dots k_{d-1}\rangle) \\
 &= \sum_{k_0=0}^{d-1} \dots \sum_{k_{d-1}=0}^{d-1} \alpha_{0k_0} \dots \alpha_{(d-1)k_{d-1}} |k_1 \dots k_{d-1} k_0\rangle \\
 &= \sum_{k_1=0}^{d-1} \dots \sum_{k_{d-1}=0}^{d-1} \sum_{k_0=0}^{d-1} \alpha_{1k_1} \dots \alpha_{(d-1)k_{d-1}} \alpha_{0k_0} |k_1 \dots k_{d-1} k_0\rangle \\
 (2.20) \quad &= |\psi_1\rangle_0 \otimes \dots \otimes |\psi_{d-1}\rangle_{d-2} \otimes |\psi_0\rangle_{d-1}
 \end{aligned}$$

as required.

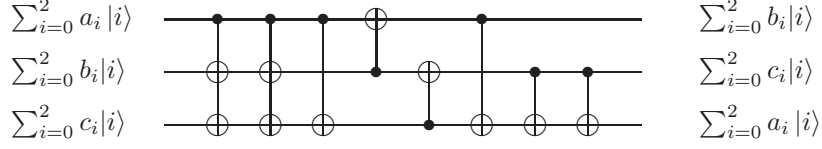


FIGURE 9. Qutrit WilNOT SWAP Network.

In particular, for an input quantum state of a d -fold quantum system whose first system \mathcal{A}_0 is prepared in the state $|e_0\rangle_0$, whose second system \mathcal{A}_1 is prepared in the state $|e_1\rangle_1$ and so forth, and whose final system \mathcal{A}_{d-1} of the input state is prepared in the state $|e_{d-1}\rangle_{d-1}$, an application of the WilNOT gate over prime dimensions yields a generalised SWAP gate so that the system \mathcal{A}_0 is in the state $|e_1\rangle_0$, the system \mathcal{A}_1 is in the state $|e_2\rangle_1$ and so forth, until the system \mathcal{A}_{d-1} is in the state $|e_0\rangle_{d-1}$. Furthermore, a WilNOT $^{(l)}$, $l < d$, operator composed of l repeating WilNOT gates can be constructed to effectuate a cyclic shift of quantum states through l quantum systems of a d -fold qudit system.

3. EXAMPLE: THE QUTRIT CASE

The qubit network that swaps two arbitrary qubit states is well known [?]. When restricted to the qubit setting, the WilNOT operator yields the unitary transformation matrices that swap the states of a pair of arbitrary qubits. We give an example of how WilNOT is used to swap the information content of three arbitrary qutrit states by defining the required unitary transformation matrices, see Fig. 9. For the case $d = 3$, the WilNOT operator produces the following sequence of states of systems $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ on input $|i\rangle_0, |j\rangle_1, |k\rangle_2$:

Stage 1. $|i\rangle_0 |j\rangle_1 |k\rangle_2$

Stage 2, step 1. $|i\rangle_0 |i+j\rangle_1 |i+j+k\rangle_2$

Stage 2, step 2. $|i\rangle_0 |2i+j\rangle_1 (|3i+2j+k\rangle_2 = |2j+k\rangle_2)$

Stage 3. $|i\rangle_0 |2i+j\rangle_1 |i+2j+k\rangle_2$

Stage 4. $(|3i+j\rangle_0 = |j\rangle_0)(|3i+3j+k\rangle_1 = |k\rangle_1 |i+2j+k\rangle_2)$

Stage 5. $|j\rangle_0 |k\rangle_1 (|i+3j+3k\rangle_2 = |i\rangle_2)$

The unitary transformation matrices associated with the WilNOT operator over $\mathbb{C}^{27} \equiv \mathbb{C}^{3^3}$ are as follows; let U_1 be the unitary transformation corresponding to Stage 1, step 1. Thus, for Stage 2, step 1 and usual lexicographic ordering for rows and columns, the matrix corresponding to U_1 is given in Fig. 10. Then $U_1(|i\rangle_0 |j\rangle_1 |k\rangle_2) = |i\rangle_0 |i+j\rangle_1 |i+j+k\rangle_2$. Let $|a\rangle_0 = \sum_{i=0}^2 a_i |i\rangle$, $|b\rangle_1 = \sum_{i=0}^2 b_i |i\rangle$, $|c\rangle_2 = \sum_{i=0}^2 c_i |i\rangle$. Then we may write $|a\rangle_0 \otimes |b\rangle_1 \otimes |c\rangle_2$ as

$$(3.1) \quad |a\rangle_0 \otimes |b\rangle_1 \otimes |c\rangle_2 = \sum_{i_1=0}^2 \sum_{i_2=0}^2 \sum_{i_3=0}^2 a_{i_1} b_{i_2} c_{i_3} |i_1 i_2 i_3\rangle.$$

Thus,

$$\begin{aligned} & U_1(|a\rangle_0 \otimes |b\rangle_1 \otimes |c\rangle_2) \\ &= U_1((a_0 |0\rangle + a_1 |1\rangle + a_2 |2\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle + b_2 |2\rangle) \otimes (c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle)) \end{aligned}$$

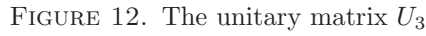
FIGURE 10. The unitary matrix U_1

The state returned after step 1, is now an entangled state. This is because there is a positive entangling power measure associated with the CNOT gate (Vatan and Williams (2004)). As we seek a generalised quantum SWAP gate, we have by

extension of the qubit SWAP gate, to construct a quantum gate whose entangling power measure is zero (Vatan and Williams (2004)). The state of the WilNOT gate remains entangled until all the unitary transformations described by WilNOT are applied. The set of unitary transformations are given by definition of each step in WilNOT. We now give the remaining set of unitary transformations and the action of each unitary on the corresponding input states.

The unitary matrix corresponding to Stage 2, step 2 is the same as U_1 . Let the transformation corresponding to Stage 2 of WilNOT be U_2 . So $U_2 = U_1^2$. We may write the action of U_2 on the state of the system prior to application of step 2 of WilNOT as

$$\begin{aligned}
& U_2(|a\rangle \otimes |b\rangle \otimes |c\rangle) \\
&= U_1(a_0b_0c_0|000\rangle + a_0b_0c_1|001\rangle + a_0b_0c_2|002\rangle + a_0b_1c_0|011\rangle + a_0b_1c_1|012\rangle \\
&\quad + a_0b_1c_2|010\rangle + a_0b_2c_0|022\rangle + a_0b_2c_1|020\rangle + a_0b_2c_2|021\rangle + a_1b_0c_0|111\rangle \\
&\quad + a_1b_0c_1|112\rangle + a_1b_0c_2|110\rangle + a_1b_1c_0|122\rangle + a_1b_1c_1|120\rangle + a_1b_1c_2|121\rangle \\
&\quad + a_1b_2c_0|100\rangle + a_1b_2c_1|101\rangle + a_1b_2c_2|102\rangle + a_2b_0c_0|222\rangle + a_2b_0c_1|220\rangle
\end{aligned}$$



Let U_3 denote the unitary transformation corresponding to Stage 3 of WilNOT. Then $U_3(|ijk\rangle) = |ij(i+k)\rangle$. The result of applying U_3 is

$$U_3(a_0b_0c_0|000\rangle + a_0b_0c_1|001\rangle + a_0b_0c_2|002\rangle + a_0b_1c_0|012\rangle + a_0b_1c_1|010\rangle \\ + a_0b_1c_2|011\rangle + a_0b_2c_0|021\rangle + a_0b_2c_1|022\rangle + a_0b_2c_2|020\rangle + a_1b_0c_0|120\rangle$$

$$\begin{aligned}
& +a_1b_0c_1|121\rangle + a_1b_0c_2|122\rangle + a_1b_1c_0|102\rangle + a_1b_1c_1|100\rangle + a_1b_1c_2|101\rangle \\
& +a_1b_2c_0|111\rangle + a_1b_2c_1|112\rangle + a_1b_2c_2|110\rangle + a_2b_0c_0|210\rangle + a_2b_0c_1|211\rangle \\
& +a_2b_0c_2|212\rangle + a_2b_1c_0|222\rangle + a_2b_1c_1|220\rangle + a_2b_1c_2|221\rangle + a_2b_2c_0|201\rangle \\
& +a_2b_2c_1|202\rangle + a_2b_2c_2|200\rangle) \\
& = a_0b_0c_0|000\rangle + a_0b_0c_1|001\rangle + a_0b_0c_2|002\rangle + a_0b_1c_0|012\rangle + a_0b_1c_1|010\rangle \\
& +a_0b_1c_2|011\rangle + a_0b_2c_0|021\rangle + a_0b_2c_1|022\rangle + a_0b_2c_2|020\rangle + a_1b_0c_0|121\rangle \\
& +a_1b_0c_1|122\rangle + a_1b_0c_2|120\rangle + a_1b_1c_0|100\rangle + a_1b_1c_1|101\rangle + a_1b_1c_2|102\rangle \\
& +a_1b_2c_0|112\rangle + a_1b_2c_1|110\rangle + a_1b_2c_2|111\rangle + a_2b_0c_0|212\rangle + a_2b_0c_1|210\rangle \\
& +a_2b_0c_2|211\rangle + a_2b_1c_0|221\rangle + a_2b_1c_1|222\rangle + a_2b_1c_2|220\rangle + a_2b_2c_0|200\rangle \\
(3.4) \quad & +a_2b_2c_1|201\rangle + a_2b_2c_2|202\rangle.
\end{aligned}$$

Let U_4 and U_5 be the unitary transformations given by $U_4(|ijk\rangle) = |(i+j)jk\rangle$ and $U_5(|ijk\rangle) = |i(j+k)k\rangle$. Then Stage 4 of WilNOT is done by applying U_4 and then U_5 . The results of applying U_4 and U_5 are

$$\begin{aligned}
& U_4(a_0b_0c_0|000\rangle + a_0b_0c_1|001\rangle + a_0b_0c_2|002\rangle + a_0b_1c_0|012\rangle + a_0b_1c_1|010\rangle \\
& +a_0b_1c_2|011\rangle + a_0b_2c_0|021\rangle + a_0b_2c_1|022\rangle + a_0b_2c_2|020\rangle + a_1b_0c_0|121\rangle \\
& +a_1b_0c_1|122\rangle + a_1b_0c_2|120\rangle + a_1b_1c_0|100\rangle + a_1b_1c_1|101\rangle + a_1b_1c_2|102\rangle \\
& +a_1b_2c_0|112\rangle + a_1b_2c_1|110\rangle + a_1b_2c_2|111\rangle + a_2b_0c_0|212\rangle + a_2b_0c_1|210\rangle \\
& +a_2b_0c_2|211\rangle + a_2b_1c_0|221\rangle + a_2b_1c_1|222\rangle + a_2b_1c_2|220\rangle + a_2b_2c_0|200\rangle \\
& +a_2b_2c_1|201\rangle + a_2b_2c_2|202\rangle) \\
& = a_0b_0c_0|000\rangle + a_0b_0c_1|001\rangle + a_0b_0c_2|002\rangle + a_0b_1c_0|112\rangle + a_0b_1c_1|110\rangle \\
& +a_0b_1c_2|111\rangle + a_0b_2c_0|221\rangle + a_0b_2c_1|222\rangle + a_0b_2c_2|220\rangle + a_1b_0c_0|021\rangle \\
& +a_1b_0c_1|022\rangle + a_1b_0c_2|020\rangle + a_1b_1c_0|100\rangle + a_1b_1c_1|101\rangle + a_1b_1c_2|102\rangle \\
& +a_1b_2c_0|212\rangle + a_1b_2c_1|210\rangle + a_1b_2c_2|211\rangle + a_2b_0c_0|012\rangle + a_2b_0c_1|010\rangle \\
& +a_2b_0c_2|011\rangle + a_2b_1c_0|121\rangle + a_2b_1c_1|122\rangle + a_2b_1c_2|120\rangle + a_2b_2c_0|200\rangle \\
(3.5) \quad & +a_2b_2c_1|201\rangle + a_2b_2c_2|202\rangle
\end{aligned}$$

and

$$\begin{aligned}
& U_5(a_0b_0c_0|000\rangle + a_0b_0c_1|001\rangle + a_0b_0c_2|002\rangle + a_0b_1c_0|112\rangle + a_0b_1c_1|110\rangle \\
& +a_0b_1c_2|111\rangle + a_0b_2c_0|221\rangle + a_0b_2c_1|222\rangle + a_0b_2c_2|220\rangle + a_1b_0c_0|021\rangle \\
& +a_1b_0c_1|022\rangle + a_1b_0c_2|020\rangle + a_1b_1c_0|100\rangle + a_1b_1c_1|101\rangle + a_1b_1c_2|102\rangle \\
& +a_1b_2c_0|212\rangle + a_1b_2c_1|210\rangle + a_1b_2c_2|211\rangle + a_2b_0c_0|012\rangle + a_2b_0c_1|010\rangle \\
& +a_2b_0c_2|011\rangle + a_2b_1c_0|121\rangle + a_2b_1c_1|122\rangle + a_2b_1c_2|120\rangle + a_2b_2c_0|200\rangle \\
& +a_2b_2c_1|201\rangle + a_2b_2c_2|202\rangle) \\
& = a_0b_0c_0|000\rangle + a_0b_0c_1|011\rangle + a_0b_0c_2|022\rangle + a_0b_1c_0|102\rangle + a_0b_1c_1|110\rangle \\
& +a_0b_1c_2|121\rangle + a_0b_2c_0|201\rangle + a_0b_2c_1|212\rangle + a_0b_2c_2|220\rangle + a_1b_0c_0|001\rangle \\
& +a_1b_0c_1|012\rangle + a_1b_0c_2|020\rangle + a_1b_1c_0|100\rangle + a_1b_1c_1|111\rangle + a_1b_1c_2|122\rangle \\
& +a_1b_2c_0|202\rangle + a_1b_2c_1|210\rangle + a_1b_2c_2|221\rangle + a_2b_0c_0|002\rangle + a_2b_0c_1|010\rangle \\
& +a_2b_0c_2|021\rangle + a_2b_1c_0|101\rangle + a_2b_1c_1|112\rangle + a_2b_1c_2|120\rangle + a_2b_2c_0|200\rangle \\
(3.6) \quad & +a_2b_2c_1|211\rangle + a_2b_2c_2|222\rangle.
\end{aligned}$$

Similarly, the action of U_7 on the state 3.7 may be given as

$$\begin{aligned}
& U_7(a_0b_0c_0|000\rangle + a_0b_0c_1|011\rangle + a_0b_0c_2|022\rangle + a_0b_1c_0|100\rangle + a_0b_1c_1|111\rangle \\
& + a_0b_1c_2|122\rangle + a_0b_2c_0|200\rangle + a_0b_2c_1|211\rangle + a_0b_2c_2|222\rangle + a_1b_0c_0|001\rangle \\
& + a_1b_0c_1|012\rangle + a_1b_0c_2|020\rangle + a_1b_1c_0|101\rangle + a_1b_1c_1|112\rangle + a_1b_1c_2|120\rangle \\
& + a_1b_2c_0|201\rangle + a_1b_2c_1|212\rangle + a_1b_2c_2|220\rangle + a_2b_0c_0|002\rangle + a_2b_0c_1|010\rangle \\
& + a_2b_0c_2|021\rangle + a_2b_1c_0|102\rangle + a_2b_1c_1|110\rangle + a_2b_1c_2|121\rangle + a_2b_2c_0|202\rangle \\
& + a_2b_2c_1|210\rangle + a_2b_2c_2|221\rangle) \\
& = a_0b_0c_0|000\rangle + a_0b_0c_1|012\rangle + a_0b_0c_2|021\rangle + a_0b_1c_0|100\rangle + a_0b_1c_1|112\rangle \\
& + a_0b_1c_2|121\rangle + a_0b_2c_0|200\rangle + a_0b_2c_1|212\rangle + a_0b_2c_2|221\rangle + a_1b_0c_0|001\rangle \\
& + a_1b_0c_1|010\rangle + a_1b_0c_2|022\rangle + a_1b_1c_0|101\rangle + a_1b_1c_1|110\rangle + a_1b_1c_2|122\rangle \\
& + a_1b_2c_0|201\rangle + a_1b_2c_1|210\rangle + a_1b_2c_2|222\rangle + a_2b_0c_0|002\rangle + a_2b_0c_1|011\rangle \\
& + a_2b_0c_2|020\rangle + a_2b_1c_0|102\rangle + a_2b_1c_1|111\rangle + a_2b_1c_2|120\rangle + a_2b_2c_0|202\rangle \\
& + a_2b_2c_1|211\rangle + a_2b_2c_2|220\rangle.
\end{aligned}
\tag{3.8}$$

FIGURE 15. The unitary matrix U_6

$$\begin{aligned}
& U_7(a_0b_0c_0|000\rangle + a_0b_0c_1|012\rangle + a_0b_0c_2|021\rangle + a_0b_1c_0|100\rangle + a_0b_1c_1|112\rangle \\
& + a_0b_1c_2|121\rangle + a_0b_2c_0|200\rangle + a_0b_2c_1|212\rangle + a_0b_2c_2|221\rangle + a_1b_0c_0|001\rangle \\
& + a_1b_0c_1|010\rangle + a_1b_0c_2|022\rangle + a_1b_1c_0|101\rangle + a_1b_1c_1|110\rangle + a_1b_1c_2|122\rangle \\
& + a_1b_2c_0|201\rangle + a_1b_2c_1|210\rangle + a_1b_2c_2|222\rangle + a_2b_0c_0|002\rangle + a_2b_0c_1|011\rangle \\
& + a_2b_0c_2|020\rangle + a_2b_1c_0|102\rangle + a_2b_1c_1|111\rangle + a_2b_1c_2|120\rangle + a_2b_2c_0|202\rangle \\
& + a_2b_2c_1|211\rangle + a_2b_2c_2|220\rangle) \\
= & a_0b_0c_0|000\rangle + a_0b_0c_1|010\rangle + a_0b_0c_2|020\rangle + a_0b_1c_0|100\rangle + a_0b_1c_1|110\rangle \\
& + a_0b_1c_2|120\rangle + a_0b_2c_0|200\rangle + a_0b_2c_1|210\rangle + a_0b_2c_2|220\rangle + a_1b_0c_0|001\rangle \\
& + a_1b_0c_1|011\rangle + a_1b_0c_2|021\rangle + a_1b_1c_0|101\rangle + a_1b_1c_1|111\rangle + a_1b_1c_2|121\rangle \\
& + a_1b_2c_0|201\rangle + a_1b_2c_1|211\rangle + a_1b_2c_2|221\rangle + a_2b_0c_0|002\rangle + a_2b_0c_1|012\rangle \\
& + a_2b_0c_2|022\rangle + a_2b_1c_0|102\rangle + a_2b_1c_1|112\rangle + a_2b_1c_2|122\rangle + a_2b_2c_0|202\rangle \\
(3.9) \quad & + a_2b_2c_1|212\rangle + a_2b_2c_2|222\rangle.
\end{aligned}$$

Note that state given in equation (3.9), the state of the system following the application of the last unitary transformation as defined by WilNOT, can be written

[illegible]

FIGURE 16. The unitary matrix U_7

as

$$\begin{aligned}
& a_0 b_0 c_0 |000\rangle + a_0 b_0 c_1 |010\rangle + a_0 b_0 c_2 |020\rangle + a_0 b_1 c_0 |100\rangle + a_0 b_1 c_1 |110\rangle \\
& + a_0 b_1 c_2 |120\rangle + a_0 b_2 c_0 |200\rangle + a_0 b_2 c_1 |210\rangle + a_0 b_2 c_2 |220\rangle + a_1 b_0 c_0 |001\rangle \\
& + a_1 b_0 c_1 |011\rangle + a_1 b_0 c_2 |021\rangle + a_1 b_1 c_0 |101\rangle + a_1 b_1 c_1 |111\rangle + a_1 b_1 c_2 |121\rangle \\
& + a_1 b_2 c_0 |201\rangle + a_1 b_2 c_1 |211\rangle + a_1 b_2 c_2 |221\rangle + a_2 b_0 c_0 |002\rangle + a_2 b_0 c_1 |012\rangle \\
& + a_2 b_0 c_2 |022\rangle + a_2 b_1 c_0 |102\rangle + a_2 b_1 c_1 |112\rangle + a_2 b_1 c_2 |122\rangle + a_2 b_2 c_0 |202\rangle \\
& + a_2 b_2 c_1 |212\rangle + a_2 b_2 c_2 |222\rangle = \\
& b_0 c_0 a_0 |000\rangle + b_0 c_0 a_1 |001\rangle + b_0 c_0 a_2 |002\rangle + b_0 c_1 a_0 |010\rangle + b_0 c_1 a_1 |011\rangle \\
& + b_0 c_1 a_2 |012\rangle + b_0 c_2 a_0 |020\rangle + b_0 c_2 a_1 |021\rangle + b_0 c_2 a_2 |022\rangle + b_1 c_0 a_0 |100\rangle \\
& + b_1 c_0 a_1 |101\rangle + b_1 c_0 a_2 |102\rangle + b_1 c_1 a_0 |110\rangle + b_1 c_1 a_1 |111\rangle + b_1 c_1 a_2 |112\rangle \\
& + b_1 c_2 a_0 |121\rangle + b_1 c_2 a_1 |121\rangle + b_1 c_2 a_2 |122\rangle + b_2 c_0 a_0 |200\rangle + b_2 c_0 a_1 |201\rangle \\
& + b_2 c_0 a_2 |202\rangle + b_2 c_1 a_0 |210\rangle + b_2 c_1 a_1 |211\rangle + b_2 c_1 a_2 |212\rangle + b_2 c_2 a_0 |220\rangle \\
& + b_2 c_2 a_1 |221\rangle + b_2 c_2 a_2 |222\rangle.
\end{aligned}
\tag{3.10}$$

The state given in equation (3.10) is separable and has the form

[illegible]

$$(3.11) \quad (b_0 |0\rangle + b_1 |1\rangle + b_2 |2\rangle) \otimes (c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle) \otimes (a_0 |0\rangle + a_1 |1\rangle + a_2 |2\rangle) = |b\rangle \otimes |c\rangle \otimes |a\rangle.$$

4. ON THE WILNOT GATE OVER EVEN DIMENSIONS GREATER THAN TWO

In this section, we consider the question of whether or not the WilNOT operator can be altered so that a generalised SWAP gate can be constructed over even dimensions. It will be shown that the answer to this question is in the negative, as we induce an unavoidable sign change in one subsystem. This question is motivated by the case $d = 4$ in which we considered if it was possible to swap the states of four 4-dimensional system whereby first system \mathcal{A}_0 prepared in the state $|e_0\rangle_0$ is left in the state $|e_1\rangle_0$, the second system \mathcal{A}_1 is prepared in the state $|e_1\rangle_1$ is left in

the state $|e_2\rangle_1$, the third system \mathcal{A}_2 prepared in the state $|e_2\rangle_2$ is left in the state $|e_3\rangle_2$ and finally the system \mathcal{A}_3 prepared in the state $|e_3\rangle_3$ is left in the state $|e_0\rangle_3$. Let us consider an operator with Stage 1 and Stage 2 identical to those of the WilNOT gate given in Section 2. In equation (2.4) (with $j = d - 1$), and at Stage 2 and step $j = d - 1$, the state of the algorithm is given by $i_0^{d-1} = i_0^0$, and $i_k^{d-1} = \sum_{m=0}^k \binom{k-m+d-2}{d-2} i_m^0$ for $k = 1, \dots, d - 1$. To effectuate the algorithm state (see equations (2.9) and (2.10))

$$(4.1) \quad \begin{pmatrix} i_0^0 \\ (d-1)i_0^0 + i_1^0 \\ i_0^0 + (d-1)i_1^0 + i_2^0 \\ \vdots \\ (d-1)i_0^0 + i_1^0 + (d-1)i_2^0 + \dots + i_{d-1}^0 \end{pmatrix}$$

on systems $\mathcal{A}_0, \dots, \mathcal{A}_{d-1}$, the WilNOT algorithm process at Stage 3 for prime d given in Section 2 requires revision when we consider dimensions $d = 0 \bmod 2$. Instead we take i_k^{d-1} with the following linear combination

$$(4.2) \quad \sum_{s=0}^{k-2} a_s i_{k-2-s}^{d-1} = \sum_{s=0}^{k-2} \left(a_s \sum_{m=0}^{(k-2)-s} \binom{(k-2)-s-m+d-2}{d-2} i_m^0 \right),$$

where

$$a_s = d - \left[\binom{s+2+d-2}{d-2} + \sum_{t=0}^{s-1} a_t \binom{s-t+d-2}{d-2} \right] + (-1)^s,$$

and the result modulo d given by (4.1) can be obtained for Stage 3, step $j = d$.

Theorem 4.1. *For $d = 0 \bmod 2$, the algorithm process at Stage 3, step $j = d$ given by*

$$\begin{aligned} i_k^d &= i_k^{d-1} + \sum_{s=0}^{k-2} a_s i_{k-2-s}^{d-1} \\ &= \sum_{m=0}^k \binom{k-m+d-2}{d-2} i_m^0 + \sum_{s=0}^{k-2} \left(a_s \sum_{m=0}^{(k-2)-s} \binom{(k-2)-s-m+d-2}{d-2} i_m^0 \right), \end{aligned}$$

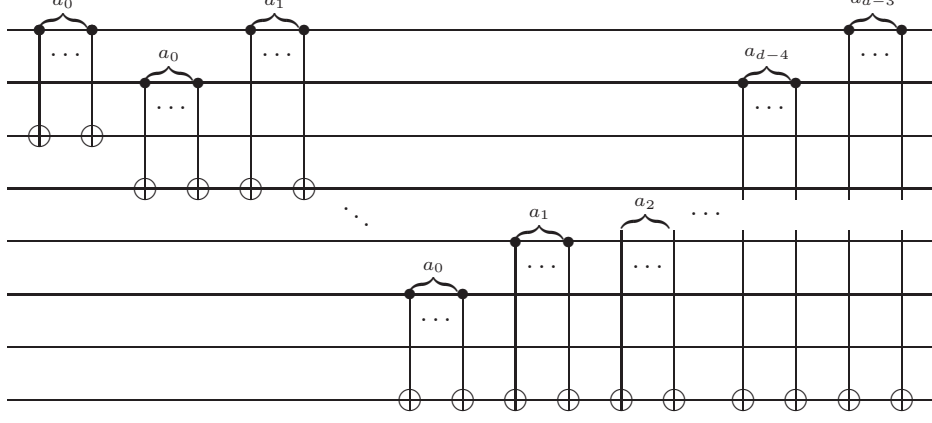
for $k = 0, \dots, d - 1$, returns outcome (4.1).

Proof. For $k = 0, 1$, we have it that $i_0^d = i_0^{d-1}$ and $i_1^d = i_1^{d-1}$. Thus, the states e_0 and e_1 are given as i_0^0 and $(d-1)i_0^0 + i_1^0$ respectively. The state e_2 is written as

$$\begin{aligned} i_2^d &= \sum_{m=0}^2 \binom{2-m+d-2}{d-2} i_m^0 + a_0 i_0^0 \\ &= \left(\binom{d}{d-2} + (d - \binom{d}{d-2} + 1) \right) i_0^0 + (d-1)i_1^0 + i_2^0 \\ (4.3) \quad &= (i_0^0 + (d-1)i_1^0 + i_2^0) \pmod{d}. \end{aligned}$$

We show by induction that, for $k = 0 \bmod 2$,

$$i_k^d = \sum_{m=0}^k \binom{k-m+d-2}{d-2} i_m^0 + \sum_{s=0}^{k-2} \left(a_s \sum_{m=0}^{(k-2)-s} \binom{(k-2)-s-m+d-2}{d-2} i_m^0 \right)$$

FIGURE 18. WilNOT gate over dimensions $d = 0 \bmod 2$; Stage 3.

$$(4.4) \quad = i_0^0 + (d-1)i_1^0 + i_2^0 + \cdots + (d-1)i_{k-1}^0 + i_k^0 \pmod{d}$$

and for $k \neq 0 \bmod 2$,

$$(4.5) \quad i_k^d = (d-1)i_0^0 + i_1^0 + (d-1)i_2^0 + \cdots + (d-1)i_{k-1}^0 + i_k^0 \pmod{d}.$$

We have shown that this is true for $k = 0, 1, 2$. Suppose $0 \leq k \leq d-2$ and further suppose that

$$\begin{aligned} i_k^d &= \sum_{m=0}^k \binom{k-m+d-2}{d-2} i_m^0 + \sum_{s=0}^{k-2} \left(a_s \sum_{m=0}^{(k-2)-s} \binom{(k-2)-s-m+d-2}{d-2} i_m^0 \right) \\ &= \sum_{m=0}^k (-1)^{k-m} i_m^0 \\ &= i_0^0 + (d-1)i_1^0 + i_2^0 + \cdots + (d-1)i_{k-1}^0 + i_k^0 \pmod{d} \end{aligned}$$

for $k = 0 \bmod 2$, and

$$= (d-1)i_0^0 + i_1^0 + (d-1)i_2^0 + \cdots + (d-1)i_{k-1}^0 + i_k^0 \pmod{d}$$

for $k \neq 0 \bmod 2$. Therefore, for $j = d$, we have,

$$i_{k+1}^d = \sum_{m=0}^{k+1} \binom{k+1-m+d-2}{d-2} i_m^0 + \sum_{s=0}^{k-1} \left(a_s \sum_{m=0}^{(k-1)-s} \binom{(k-1)-s-m+d-2}{d-2} i_m^0 \right)$$

$$\begin{aligned}
&= \sum_{m=0}^k \left(\binom{k-m+d-2}{d-2} i_{m+1}^0 + \binom{k+1+d-2}{d-2} i_0^0 \right) \\
&\quad + \sum_{s=0}^{k-1} a_s \left(\left(\sum_{m=0}^{(k-2)-s} \binom{(k-2)-s-m+d-2}{d-2} i_{m+1}^0 \right) + \binom{(k-1)-s+d-2}{d-2} i_0^0 \right) \\
&= \sum_{m=0}^k (-1)^{k-m} i_{m+1}^0 + \left(\binom{k+1+d-2}{d-2} + \sum_{s=0}^{k-1} a_s \binom{k-1-s+d-2}{d-2} \right) i_0^0.
\end{aligned}$$

Recall that the binomial coefficients of $i_k^{d-1} = \sum_{m=0}^k \binom{k-m+d-2}{d-2} i_m^0$ are precisely those coefficients of $i_{k+1}^{d-1} = \sum_{m=0}^{k+1} \binom{k+1-m+d-2}{d-2} i_m^0$ for $m = 1, \dots, k+1$. Hence, the particular combination of systems $\mathcal{A}_{(k-2)-s}$ that return the state $i_k^d = (i_0^0 + (d-1)i_1^0 + i_2^0 + \dots + (d-1)i_{k-1}^0 + i_k^0) \bmod d$ is the combination that effectuates a similar sequence on i_{k+1}^d for $m = 1, \dots, k+1$. Since $k = 0 \bmod 2$, then for i_{k+1}^d we require that the scalar value for i_0^0 degenerates to $(d-1) \bmod d$. Thus, for $m = 0$ and by definition of a_s , we have

$$\begin{aligned}
&\left(\binom{(k+1)+d-2}{d-2} + \sum_{s=0}^{k-1} a_s \binom{(k-1)-s+d-2}{d-2} \right) i_0^0 \\
&= \left(\binom{(k+1)+d-2}{d-2} + \sum_{s=0}^{k-2} \left(a_s \binom{(k-1)-s+d-2}{d-2} \right) + a_{k-1} \right) i_0^0 \\
&= \left(\binom{(k+1)+d-2}{d-2} + \sum_{s=0}^{k-2} a_s \binom{(k-1)-s+d-2}{d-2} \right. \\
&\quad \left. + \left(d - \left[\binom{(k+1)+d-2}{d-2} + \sum_{s=0}^{k-2} a_s \binom{(k-1)-s+d-2}{d-2} \right] + (-1)^{k+1} \right) \right) i_0^0 \\
&= (-1)^{k+1} i_0^0.
\end{aligned}$$

Hence, $i_{k+1}^d = \sum_{m=0}^{k+1} (-1)^{k+1-m} i_m^0$, and the result follows. We now implement Stage 4 of the WilNOT gate, which is written as

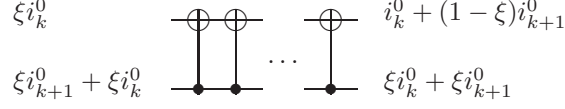
$$(4.6) \quad i_k^{d+1} = i_k^d + i_{k+1}^d$$

for $k = 0, \dots, d-2$ and

$$(4.7) \quad i_{d-1}^{d+1} = i_{d-1}^d + \sum_{m=0}^{d-2} (-1)^{d-1-s} i_m^0,$$

and a revised Stage 5 given by $\sum_{k=1}^{d-1} \eta_k^* i_k^{d+2} = \sum_{t=0}^{\lfloor \frac{d-1}{2} \rfloor} (d-1) i_{2t+1}^0 + \sum_{t=0}^{\frac{d-2}{2}} i_{2t}^0$ which then returns

$$(4.8) \quad \begin{pmatrix} i_1^0 \\ i_2^0 \\ i_3^0 \\ \vdots \\ i_{d-1}^0 \\ (d-1)i_0^0 \end{pmatrix}.$$

FIGURE 19. $P_{\xi} - 1$ gates on pairs (i_k, i_{k+1}) .

Although we have not achieved our aim, our modification of the WilNOT gate for d even has produced a similar state, namely a SWAP but with a sign change for the system \mathcal{A}_{d-1} . We have not been able to modify WilNOT to produce the SWAP gate. We give the following argument to show that a different approach would be required. Result (4.8) is a particular outcome for an even valued d and once entered into this sequence of CNOTs seems not to return an output with scalars on e_0, \dots, e_{d-1} all equal to unity. To show this claim, let us consider the more general case given by

$$(4.9) \quad \begin{pmatrix} \xi i_1^0 \\ \xi i_2^0 \\ \xi i_3^0 \\ \vdots \\ \xi i_{d-1}^0 \\ (d - \xi) i_0^0 \end{pmatrix}.$$

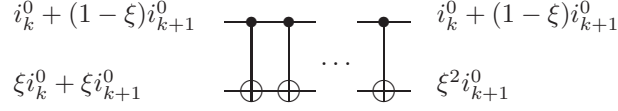
Consider the pairs $(\xi i_k^0, \xi i_{k+1}^0)$, for $k \in \{1, \dots, d-3\}$, and further consider the pair $(\xi i_{d-1}^0, \xi i_0^0)$. Given the paired input sequence $(\xi i_k^0, \xi i_{k+1}^0)$, for $k \in \{1, \dots, d-3\}$, and a mapping that targets i_{k+1} , we have it that $(\xi i_k^0, \xi i_{k+1}^0) \mapsto (\xi i_k^0, \xi i_k^0 + \xi i_{k+1}^0)$. Denote by P_{ξ} the inverse mod d of ξ , whence, $P_{\xi}\xi = 1 \pmod{d}$. Applying $P_{\xi} - 1$ gates, see Figure 19, from the control with value $\xi i_k^0 + \xi i_{k+1}^0$ to the target corresponding to the output ξi_k^0 results in

$$(4.10) \quad \begin{aligned} (\xi i_k^0, \xi i_k^0 + \xi i_{k+1}^0) &\mapsto (P_{\xi}\xi i_k^0 + (P_{\xi} - 1)\xi i_{k+1}^0, \xi i_k^0 + \xi i_{k+1}^0) \\ &= (i_k^0 + (1 - \xi)i_{k+1}^0, \xi i_k^0 + \xi i_{k+1}^0). \end{aligned}$$

To eliminate the value ξi_k^0 from result (4.10), $d - \xi$ gates are implemented on the target $\xi i_k^0 + \xi i_{k+1}^0$ thereby illustrating the mapping

$$(4.11) \quad \begin{aligned} (i_k^0 + (1 - \xi)i_{k+1}^0, \xi i_k^0 + \xi i_{k+1}^0) &\mapsto (i_k^0 + (1 - \xi)i_{k+1}^0, \xi i_k^0 + \xi i_{k+1}^0 \\ &\quad + (d - \xi)(i_k^0 + (1 - \xi)i_{k+1}^0)) \\ &= (i_k^0 + (1 - \xi)i_{k+1}^0, \xi i_{k+1}^0 \\ &\quad + (d - \xi)(1 - \xi)i_{k+1}^0) \\ &= (i_k^0 + (1 - \xi)i_{k+1}^0, \xi i_{k+1}^0 \\ &\quad + (-\xi + \xi^2)i_{k+1}^0) \\ &= (i_k^0 + (1 - \xi)i_{k+1}^0, \xi^2 i_{k+1}^0). \end{aligned}$$

In a similar fashion, let us consider the final pair $(\xi i_{d-1}, (d - \xi)i_0)$. Applying those gates that correspond to result (4.10) and result (4.11) on the pair $\xi i_{d-1}, (d - \xi)i_0$

FIGURE 20. $d - \xi$ gates on (i_{d-2}, i_{d-1}) .

returns the outcome $(i_{d-1}^0 + (\xi - 1)i_0^0, -\xi^2 i_0^0)$. Thus, we have the mapping given by

$$(4.12) \quad \begin{pmatrix} \xi i_1^0 \\ \xi i_2^0 \\ \vdots \\ \xi i_{d-3}^0 \\ \xi i_{d-2}^0 \\ \xi i_{d-1}^0 \\ (d - \xi)i_0^0 \end{pmatrix} \mapsto \begin{pmatrix} i_1^0 + (1 - \xi)i_2^0 \\ \xi^2 i_2^0 \\ \vdots \\ i_{d-3}^0 + (1 - \xi)i_{d-2}^0 \\ \xi^2 i_{d-2}^0 \\ i_{d-1}^0 + (\xi - 1)i_0^0 \\ -\xi^2 i_0^0 \end{pmatrix}.$$

Since the scalar values ξ^2 and $-\xi^2$ can not simultaneously be one, it seems that any mapping on the state (4.9) will fail to return a state whose scalar values all equal unity. That such outcome in result (4.12) is best possible suggests that the WilNOT algorithm fails to extend over dimensions $d = 0 \pmod{2}$. Therefore, it seems that WilNOT cannot be modified for the case d even to permit a cycle of states such that first system \mathcal{A}_0 prepared in the state $|e_0\rangle_0$ is left in the state $|e_1\rangle_0$, the second system \mathcal{A}_1 prepared in the state $|e_1\rangle_1$ is left in the state $|e_2\rangle_1$, the third system \mathcal{A}_2 prepared in the state $|e_2\rangle_2$ is left in the state $|e_3\rangle_2$ and finally the system \mathcal{A}_3 prepared in the state $|e_3\rangle_3$ is left in the state $|e_0\rangle_3$. Interestingly, WilNOT can demonstrate the case where first system \mathcal{A}_0 prepared in the state $|e_0\rangle_0$ is left in the state $|e_2\rangle_0$, the second system \mathcal{A}_1 prepared in the state $|e_1\rangle_1$ is left in the state $|e_3\rangle_1$, the third system \mathcal{A}_2 prepared in the state $|e_2\rangle_2$ is left in the state $|e_0\rangle_2$ and finally the system \mathcal{A}_3 prepared in the state $|e_3\rangle_3$ is left in the state $|e_1\rangle_3$.

5. CONCLUSION

A construction generalising the qubit SWAP gate to qudit quantum system has been presented. This construction is composed entirely from instances of the CNOT gate. Over prime dimensions d , our construction cycles the states of d qudits while over even valued dimensions $d > 2$, we show that our construction can not be modified to permit a cycle of states.

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